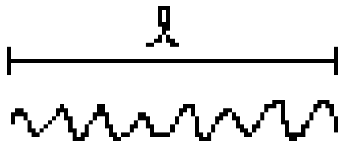
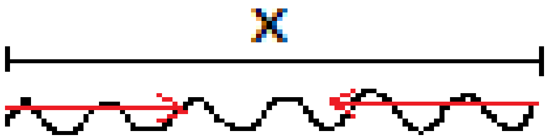
A.1 Springs

Any elastic substance will exert a spring-like force to good approximation. Shock absorbers in car, cushioning in shoes, rubber bands, actual springs, pillows, even ostensibly hard surfaces like steel block are elastic, technically.



And an elastic substance can be characterized by two things: its equilibrium length, ℓ, and its stiffness (spring constant) k. Obviously k for a rubber band is much less than k for a steel rod.

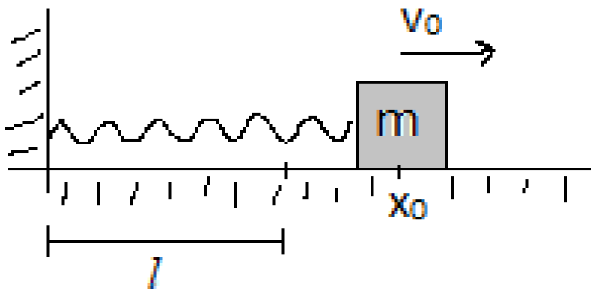


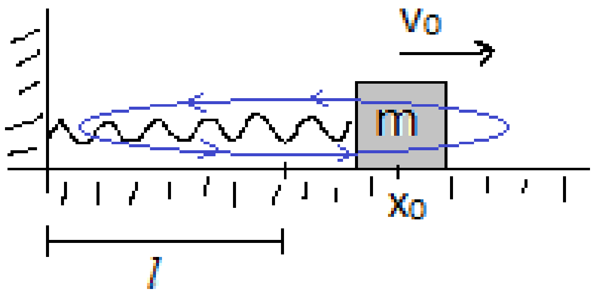
$$F_s = k(x - \ell)$$

If an elastic substance is stretched (or compressed) to a new length x, then it will exert a 'restoring' force proportional to that stretch or compression. The proportionality constant is of course, k.

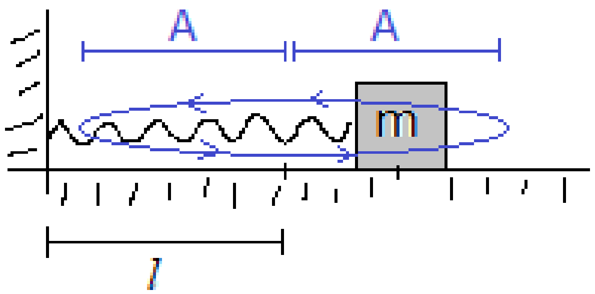
A.1 Springs

Now suppose that we attach a mass m to the elastic substance (henceforth just referred to as the ‘spring’), stretch it out to some initial length x_0, and release it at some initial speed v_0. What will it do? Can we predict where it will be at any time thereafter?





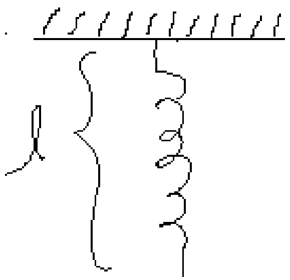
We’d suspect it to oscillate back and forth about its equilibrium position. This is called *harmonic* motion.



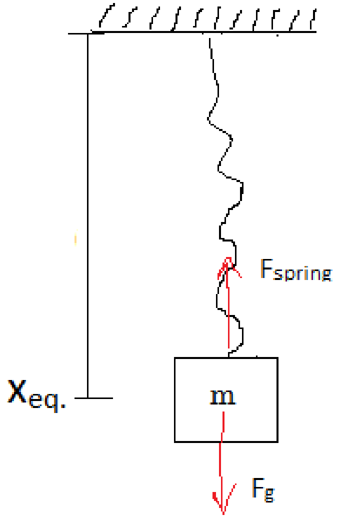
The maximum displacement from its equilibrium position we call the ‘amplitude’ A. The time it takes to make a complete oscillation is called the period, T.

A.1 Springs

Let’s consider a different setup, where we have a *hanging* spring, instead of a horizontal one.



With nothing on the spring it will assume its equilibrium length

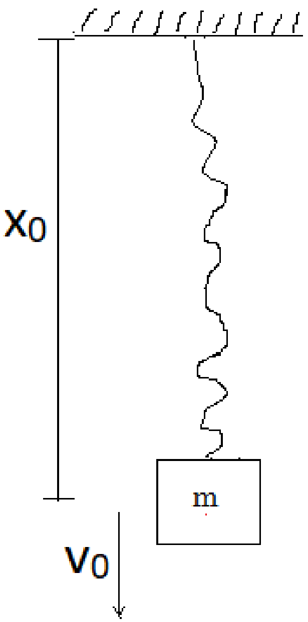


When we place a mass on the spring, it will assume a new equilibrium length, where the forces balance.

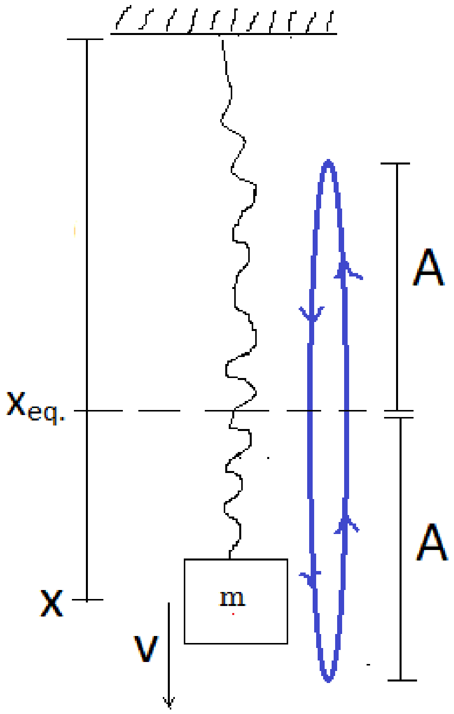
$$\sum F_x = 0$$

$$mg - k(x_{eq.} - l) = 0$$

$$x_{eq.} = l + \frac{mg}{k}$$



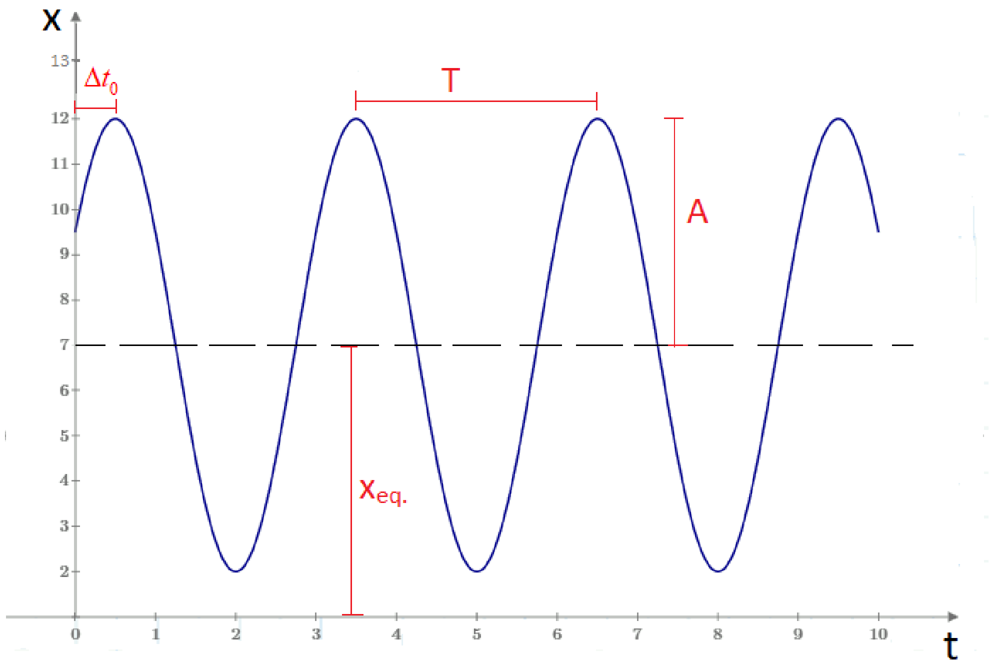
Then we may give it an initial displacement and initial velocity and see what it does.



Again, we’d suspect harmonic oscillation about the equilibrium point, with some amplitude A and period T.

A.1 Springs

But again, can we predict where it will be at any given time? To do so, we need a mathematical formula that describes harmonic motion. There are lots of varieties of harmonic motion, and lots of mathematical functions to go along with them, but the simplest variety of harmonic motion, i.e., simple harmonic motion (SHM), can be described with trig functions.



$$x(t) = x_{eq.} + A\cos(\omega t + \varphi_0)$$

$x_{eq.}$ = point about which the oscillation occurs.

A = 'amplitude' = max displacement about $x_{eq.}$

T = 'period' = time for one complete oscillation

f = 'frequency' = rate of oscillation $= \dfrac{1}{T}$ [unit = Hz]

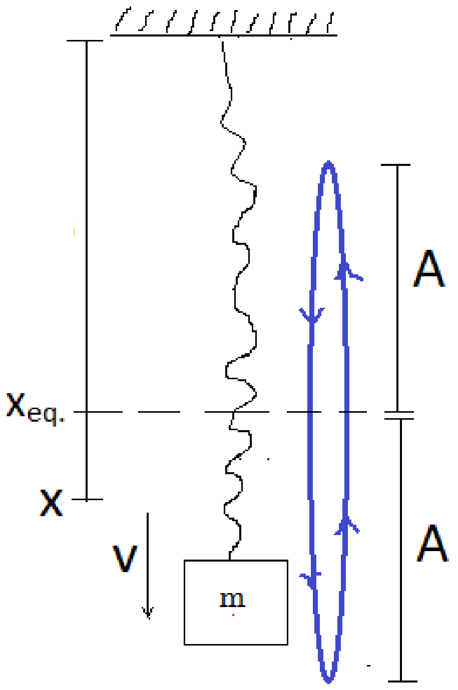
ω = 'angular frequency' $= 2\pi f$

φ_0 = 'phase constant' = # of radians cos peak is offset from origin

$= \dfrac{2\pi}{T}\Delta t_0$ (note Δt_0 is negative/positive to right/left of origin)

A.1 Springs

Now we have to see if we're lucky, and that simple harmonic motion is the kind of harmonic motion a mass on a spring undergoes. To ascertain, we'll see if our formula accords with Newton's second law's description of the motion. I'll just analyze the hanging spring example – the other cases are similar. According to N2L we will have:



$$\sum F_x = ma_x$$

$$mg - k(x-l) = ma_x$$

$$mg - k(x-l) = m\frac{d^2x}{dt^2}$$

$$mg - k\left[\left(l + \frac{mg}{k} + A\cos(\omega t + \varphi_0)\right) - l\right] = m\frac{d^2}{dt^2}\left(l + \frac{mg}{k} + A\cos(\omega t + \varphi_0)\right)$$

And now we plug our proposed x(t) into N2L and see if it is consistent

$$-k\left[A\cos(\omega t + \varphi_0)\right] = m \cdot -\omega^2 A\cos(\omega t + \varphi_0)$$

$$k = m\omega^2$$

$$\omega = \sqrt{\frac{k}{m}}$$

It is! with the stipulation that omega is given by this value.

A.1 Springs

If we tested the other cases: standing spring, horizontal spring, we would find the same conclusion: that simple harmonic motion is the correct description of their motion, and that the angular frequency is given by ω = √(k/m). To summarize, then,

$$x(t) = x_{eq.} + A\cos(\omega t + \varphi_0)$$

$x_{eq.} = l, l + \dfrac{mg}{k}, etc.,$ is the position at which the forces balance.

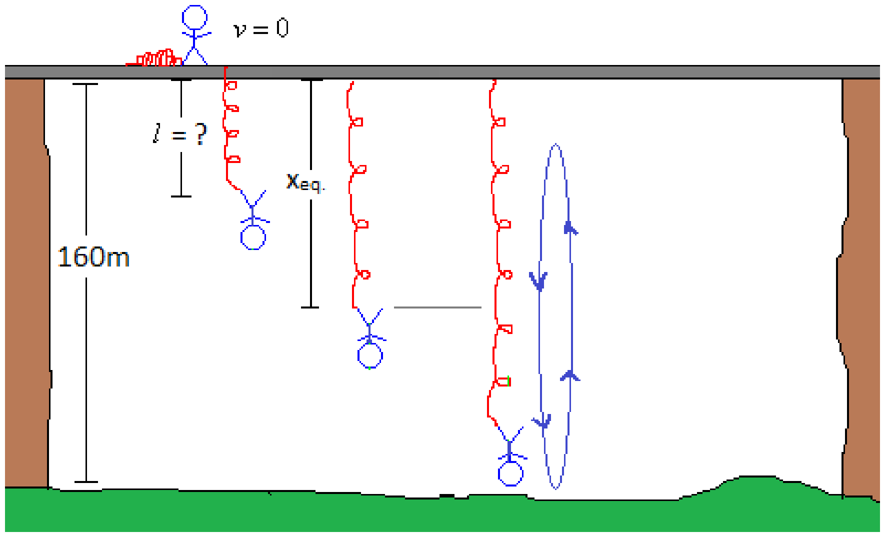
$A =$ unknown as yet, must be determined by initial conditions or energy considerations

$$\omega = \sqrt{\frac{k}{m}} \longrightarrow f = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \longrightarrow T = 2\pi\sqrt{\frac{m}{k}}$$

$\varphi_0 =$ unknown as yet, must be determined by initial conditions or energy considerations

A.1 Springs

Suppose we have an m = 70kg person jumping off a bridge with a k = 40N/m spring constant bungee cord cut to some unspecified length ℓ. We want the length to be just right so that the person reaches the bottom of the canyon just at the moment he comes to rest, before moving back upwards.



(a) To what length should the cord be cut so that they reach the bottom with zero velocity?

$$E_{top} = E_{bottom}$$

$$\frac{1}{2}mv_{top}^2 + mgy_{top} + \frac{1}{2}k(x_{top} - l)^2 = \frac{1}{2}mv_{bottom}^2 + mgy_{bottom} + \frac{1}{2}k(x_{bottom} - l)^2$$

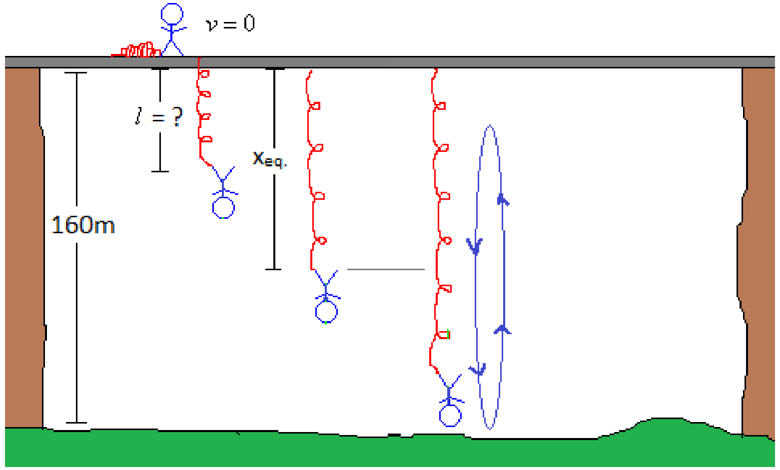
$$\frac{1}{2}m(0) + mg(160) + \frac{1}{2}k(l-l)^2 = \frac{1}{2}m(0) + mg(0) + \frac{1}{2}k(160-l)^2$$

$$mg(160) = \frac{1}{2}k(160-l)^2$$

$$\sqrt{\frac{2mg(160)}{k}} = (160-l)$$

$$l = 160 - \sqrt{\frac{2mg(160)}{k}} = 160 - \sqrt{\frac{2(70)(9.8)(160)}{40}} = 86$$

A.1 Springs



(b) Next we want to know equilibrium point about which he'll oscillate?

$$\sum F_y = 0$$

This is where the forces balance...

$$-k(x-l)+mg=0$$

$$x = l + \frac{mg}{k} = 86 + \frac{(70)(9.8)}{40} = 103\text{m}$$

(c) And the period of oscillation?

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{k/m}} = \frac{2\pi}{\sqrt{40/70}} = \frac{2\pi}{0.76\,\text{rad/s}} = 8.3\text{s}$$

(d) Now we'd like to know the fully specified equation for the person's motion...

So far we know x(t) = $x_{eq.}$ + Acos(ωt + ϕ_0) = 104 + Acos(0.76t + ϕ_0). To get A and ϕ_0, we need to know the initial conditions, i.e. the person's initial position and velocity, at the point when the loose cord starts acting like a spring. This is when it has fully unraveled to ℓ = 86m. So 86m is the initial position x_0. To get the initial velocity, we can use energy conservation....

$$E_{top} = E_l \longrightarrow \frac{1}{2}mv_{top}^2 + mgy_{top} + \frac{1}{2}k(x_{top}-l)^2 = \frac{1}{2}mv_l^2 + mgy_l + \frac{1}{2}k(x_l-l)^2$$

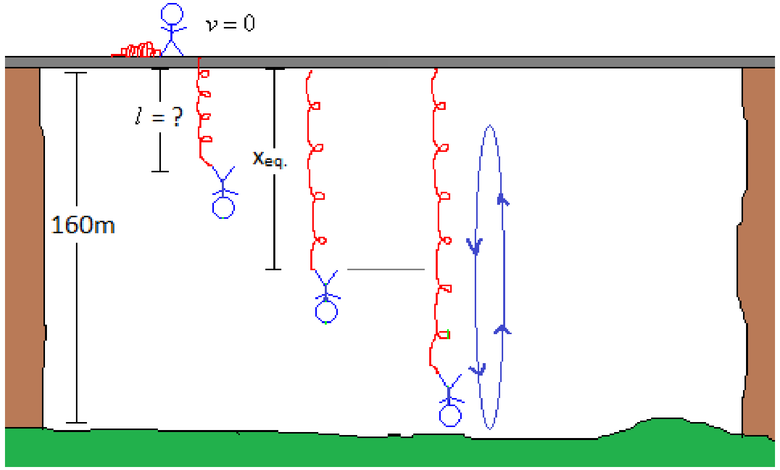
$$\frac{1}{2}m(0) + mg(160) + \frac{1}{2}k(l-l)^2 = \frac{1}{2}mv_l^2 + mg(160-l) + \frac{1}{2}k(l-l)^2$$

$$mg(160) = \frac{1}{2}mv_l^2 + mg(160-86)$$

$$mg(86) = \frac{1}{2}mv_l^2$$

$$v_l = \sqrt{2g(86)} = 41\text{m/s}$$

A.1 Springs



And this is how we use the initial conditions to get the amplitude and phase constant…

(1)

$$x(t)\big|_{t=0} = x_0$$

$$104 + A\cos(0.76t + \varphi_0)\big|_{t=0} = 86$$

$$A\cos\varphi_0 = -18$$

(2)

$$v(t)\big|_{t=0} = v_0$$

$$\left.\frac{dx}{dt}\right|_{t=0} = 41$$

$$-0.76A\sin(0.76t + \varphi_0)\big|_{t=0} = 41$$

$$-0.76A\sin(\varphi_0) = 41$$

$$A\sin\varphi_0 = -54$$

Now we have two equations and two unknowns. And this is how we can solve them to get A and ϕ_0…

$$(1)^2 + (2)^2 \rightarrow$$

$$A^2\cos^2\varphi_0 + A^2\sin^2\varphi_0 = (-18)^2 + (-54)^2$$

$$A^2(\cos^2\varphi_0 + \sin^2\varphi_0) = (18)^2 + (54)^2$$

$$A^2(1) = 18^2 + 54^2$$

$$A = \sqrt{18^2 + 54^2} = 57$$

and

$$\frac{(2)}{(1)} \rightarrow$$

$$\frac{A\sin\varphi_0}{A\cos\varphi_0} = \frac{-54}{-18}$$

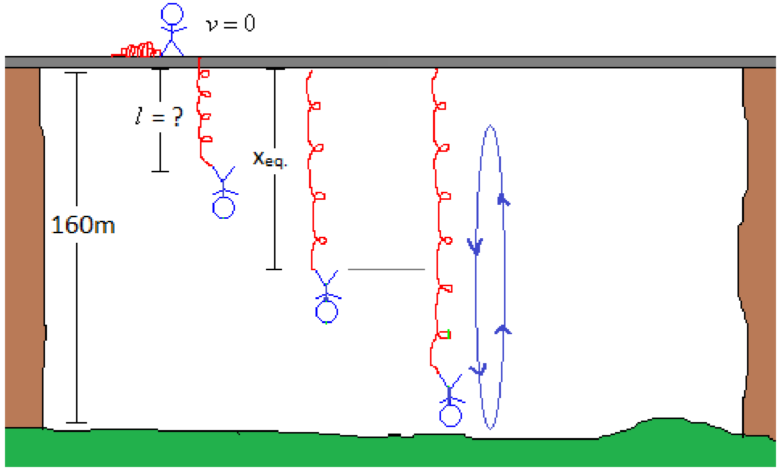
$$\tan\varphi_0 = \frac{-54}{-18}$$

$$\varphi_0 = \tan^{-1}\left(\frac{-54}{-18}\right) = \pi + \tan^{-1}\left(\frac{54}{18}\right) = 4.4\,\text{rad}$$

so

$$x(t) = 103 + 57\cos(0.76t + 4.4)$$

A.1 Springs



(e) what is the position, velocity, acceleration of our intrepid bungee jumper at time t = 12s?

So we just evaluate our x(t) equation, and its first and second derivative, at this time.

$$x(12) = 103 + 57\cos(0.76t + 4.4)\Big|_{t=12}$$
$$= 135\,\text{m}$$

$$v(12) = \frac{dx}{dt}\Big|_{t=12}$$
$$= -0.76 \cdot 57\sin(0.76t + 4.4)\Big|_{t=12}$$
$$= -43.3\sin(0.76t + 4.4)\Big|_{t=12}$$
$$= -35\,\text{m/s}$$

$$a(12) = \frac{d^2x}{dt^2}\Big|_{t=12}$$
$$= -0.76^2 \cdot 57\cos(0.76t + 4.4)\Big|_{t=12}$$
$$= -33\cos(0.76 \cdot 12 + 4.4)$$
$$= -19\,\text{m/s}^2$$

(f) What is the lowest and highest position our person will attain? What will be his maximum speed? Maximum acceleration?

max/min x is when cos = ±1

$$x(t)_{\text{max/min}} = 103 \pm 57$$
$$= 160\text{m}, 46\text{m}$$

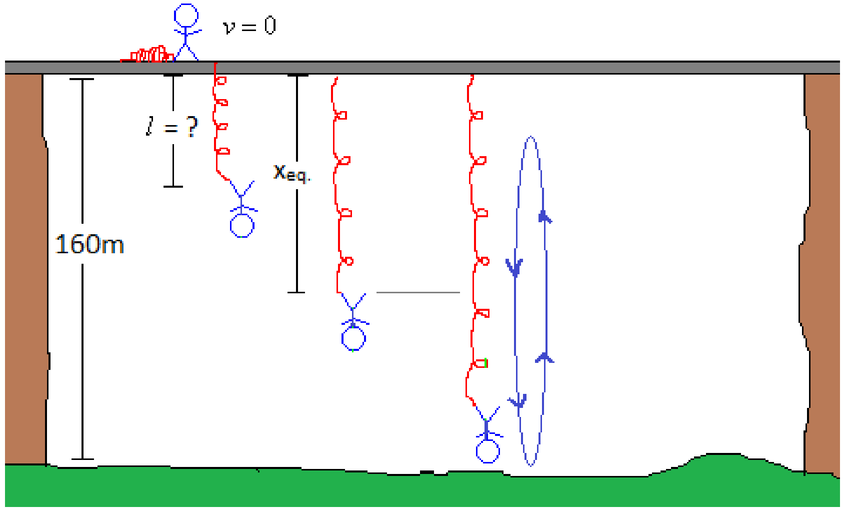
max v is when cos = -1

$$v(t)_{\text{max}} = 43.3\text{m/s}$$

max a is when cos = -1

$$a(t)_{\text{max}} = 33\,\text{m/s}^2$$

A.1 Springs



(g) If the maximum tension the cord can withstand is T = 2500N, will the cord snap?

To get the force the cord exerts, we have to use N2L

$$\sum F_y = ma_y$$
$$mg - T = ma_y$$
$$T = mg - ma_y$$

The tension will be maximum when a = -33m/s^2, in which case we'd have:

$$T_{max} = (70\,\text{kg})(9.8\text{m/s}^2) - (70\,\text{kg})(-33\text{m/s}^2)$$
$$= 3000\text{N}$$

Alas...

(h) When will the cord snap?

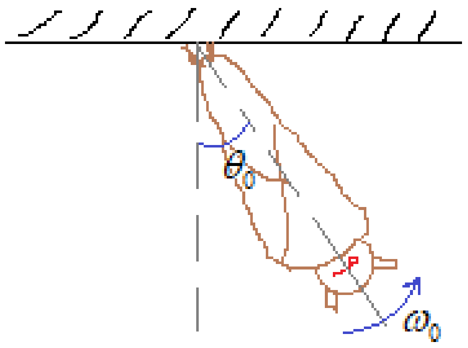
To ascertain, we'd plug our expression for a_y into the tension equation, and set T = 2500.

$$T = mg - ma_y$$
$$2500 = (70)(9.8) - (70)\left[-33\cos(0.76t + 4.4)\right]$$
$$\cos(0.76t + 4.4) = 0.785$$
$$0.76t + 4.4 = \cos^{-1}(0.785)$$

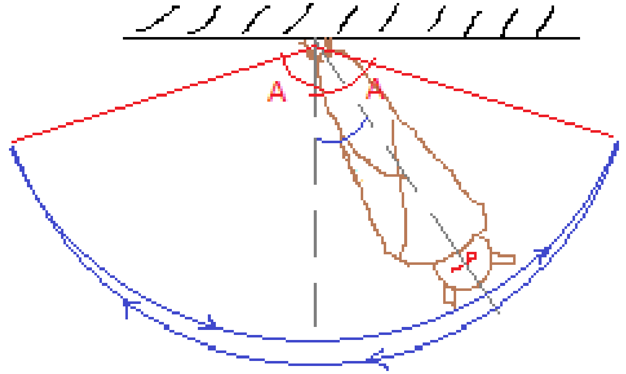
$$t = \frac{-4.4 + \cos^{-1}(0.785)}{0.76}$$
$$= \frac{-4.4 + (2\pi - 0.67)}{0.76}$$
$$= 1.6\text{s}$$

acos(0.785) has multiple values and I'm using the one which gives the first positive time

A.2 Pendula



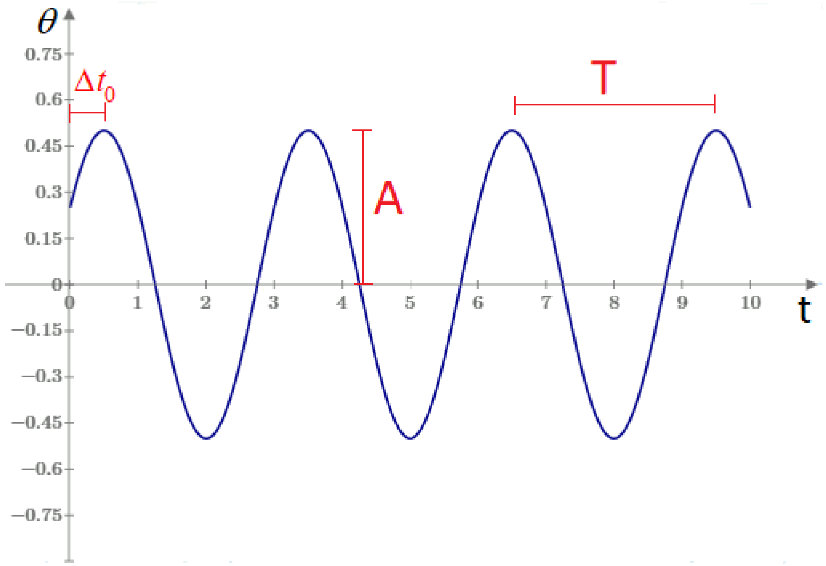
Say we've got a bat tilted to an initial angle Θ_0, and given an initial angular velocity ω_0. What would its motion look like?



We would expect, of course, that it will oscillate about it's equilibrium point, $\Theta = 0$, with amplitude A (note A is an *angle*), and period T. So again, it will engage in *harmonic* motion.

A.2 Pendula

We'd like to be able to predict what A and T will be, and moreover, to precisely predict the angular position of the bat at any time after t = 0. So we need a mathematical function that describes harmonic motion, and we'll turn to our *simple* harmonic function again.



$$\theta(t) = A\cos(\omega t + \varphi_0)$$

A = 'amplitude' = max angular displacement about 0°

T = 'period' = time for one complete oscillation

f = 'frequency' = rate of oscillation $= \dfrac{1}{T}$ [unit = Hz]

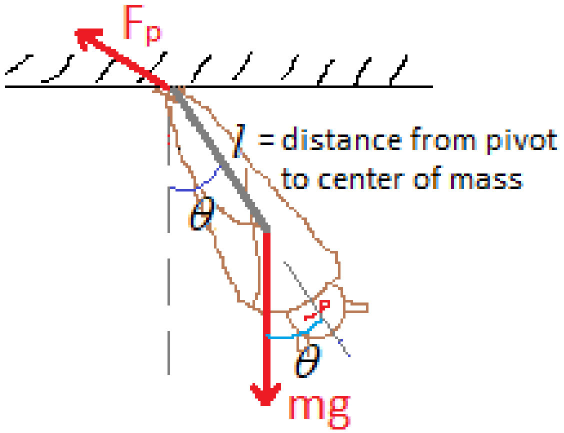
ω = 'angular frequency' $= 2\pi f$

φ_0 = 'phase constant' = # of radians cos peak is offset from origin

$$= \frac{2\pi}{T}\Delta t_0$$ (note Δt_0 is negative/positive to right/left of origin)

A.2 Pendula

Now we have to see if we're lucky, and that simple harmonic motion is the kind of harmonic motion a 'pendulum' undergoes. To ascertain, we'll see if our formula accords with Newton's second law's description of the motion. According to N2L we will have:



$$\sum \tau = I\alpha$$

$$\cancel{\tau_{F_p}} + \tau_{mg} = I\alpha$$

$$-lmg\sin\theta = I\frac{d^2\theta}{dt^2}$$

$$-lmg\sin\left[A\cos(\omega t+\varphi_0)\right] = I\frac{d^2}{dt^2}A\cos(\omega t+\varphi_0)$$

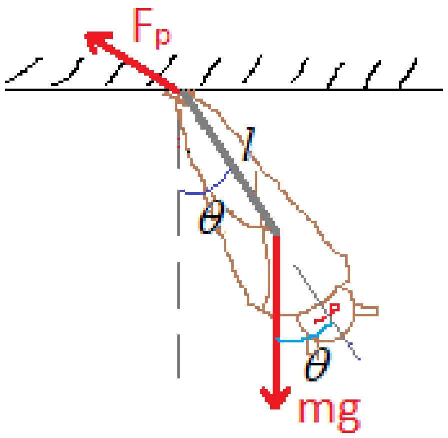
And now we plug our proposed Θ(t) into N2L and see if it is consistent

$$-lmg\sin\left[A\cos(\omega t+\varphi_0)\right] = I\cdot -\omega^2 A\cos(\omega t+\varphi_0)$$

$$\frac{\sin\left[A\cos(\omega t+\varphi_0)\right]}{A\cos(\omega t+\varphi_0)} = \frac{I\omega^2}{mgl}$$

This doesn't work!` On the LHS we have a function of t, which changes with time, and on the RHS we have a constant. Something that changes can't be perpetually equal to something that doesn't.

A.2 Pendula



So this means that while the oscillatory motion of a pendulum *is* harmonic motion, it is *not* simple harmonic motion. So the mathematical formula that describes the motion is *not* $\Theta(t) = A\cos(\omega t + \phi_0)$. Rather it is some more complicated function that *has* been derived, but which takes us too far outside the scope of this course. We'll settle for the following accommodation, called the ***small angle approximation***.

small angle approximation: if $\theta < 30^\circ$ or so, then $\sin\theta \approx \theta$

Θ (rad)	sin(Θ)
0.1 (≈6°)	0.10
0.3 (≈17°)	0.30
0.5 (≈29°)	0.48
0.7 (≈40°)	0.64
0.9 (≈52°)	0.78
1.1 (≈63°)	0.90
1.3 (≈74°)	0.96
1.5 (≈86°)	1.00

Under this condition, we can simplify our derivation's last line:

$$\frac{\sin\left[A\cos(\omega t+\varphi_0)\right]}{A\cos(\omega t+\varphi_0)} = \frac{I\omega^2}{mgl}$$

$$\frac{\left[A\cos(\omega t+\varphi_0)\right]}{A\cos(\omega t+\varphi_0)} \approx \frac{I\omega^2}{mgl}$$

$$1 \approx \frac{I\omega^2}{mgl}$$

$$\omega \approx \sqrt{\frac{mgl}{I}}$$

A.2 Pendula

To summarize, then, we find that for small angles (less than 30 degrees or so), the harmonic motion of the pendulum can be well approximated as *simple* harmonic motion, described by the following formula:

$$\theta(t) = A\cos(\omega t + \varphi_0)$$

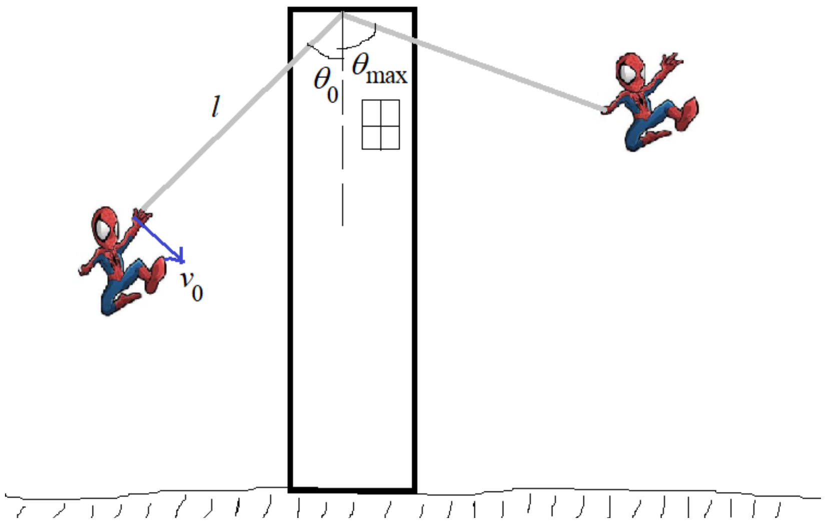
$A =$ unknown as yet, must be determined by initial conditions or energy considerations

$$\omega = \sqrt{\frac{mgl}{I}} \longrightarrow f = \frac{1}{2\pi}\sqrt{\frac{mgl}{I}} \longrightarrow T = 2\pi\sqrt{\frac{I}{mgl}}$$

$\varphi_0 =$ unknown as yet, must be determined by initial conditions or energy considerations

A.2 Pendula

Let's say that Spiderman (m = 60kg) is swinging on a 30m long (2kg) web. He starts at a 40 degree angle and a speed of 20m/s.
(a) What is his angular position as a function of time? (can use simple harmonic approximation, even though the angles are outside the range of validity of our simple harmonic motion formula)



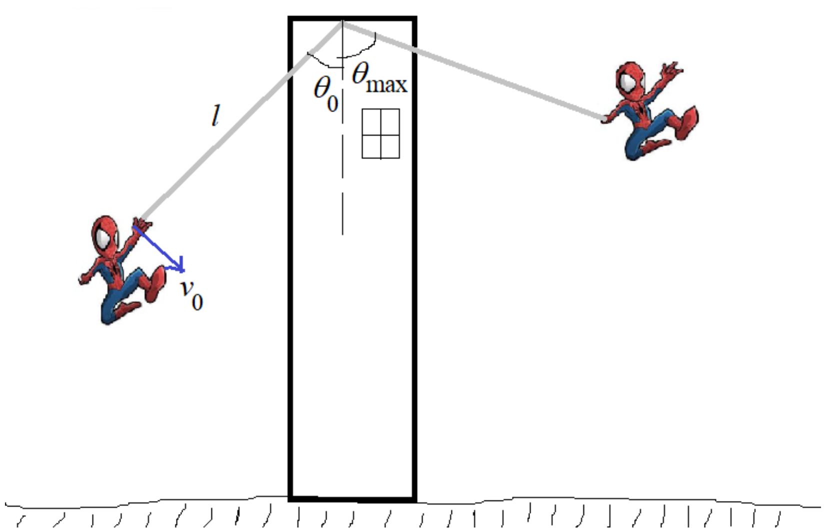
Let's get the angular frequency first. To do so, we need to get the moment of inertia and center of mass.

$$I = I_{sm} + I_{web}$$
$$= m_{sm}l^2 + \frac{1}{3}m_{web}l^2$$
$$= (60\,\text{kg})(30\text{m})^2 + \frac{1}{3}(2\,\text{kg})(30\text{m})^2$$
$$= 54600\,\text{kg}\cdot\text{m}^2$$

$$l = \frac{m_{sm}r_{sm} + m_{web}r_{web}}{m_{sm} + m_{web}}$$
$$= \frac{(60\,\text{kg})(30\text{m}) + (2\,\text{kg})(15\text{m})}{60\,\text{kg} + 2\,\text{kg}}$$
$$= 29.5\text{m}$$

So then, $\omega = \sqrt{\frac{mgl}{I}} = \sqrt{\frac{(62\,\text{kg})(9.8\text{m/s}^2)(29.5\text{m})}{(54600\,\text{kg}\cdot\text{m}^2)}} = 0.57\,\text{rad/s}$

A.2 Pendula



Next we incorporate the initial conditions…

(1)

$$\theta(0)=\theta_0$$

$$A\cos(0.57t+\varphi_0)\big|_{t=0} = -40^\circ \cdot \frac{\pi}{180}$$

$$A\cos\varphi_0 = -0.7\,\mathrm{rad}$$

(2)

$$\omega(0)=\omega_0$$

$$\left.\frac{d\theta}{dt}\right|_{t=0} = +\frac{v_0}{l}$$

$$-0.57A\sin(0.57t+\varphi_0)\big|_{t=0} = \frac{20\mathrm{m/s}}{30\mathrm{m}}$$

$$-0.57A\sin\varphi_0 = 2/3$$

$$A\sin\varphi_0 = -1.18$$

And do the usual, to get A and ϕ_0.

$$(1)^2+(2)^2$$

$$A^2\cos^2\varphi_0 + A^2\sin^2\varphi_0 = (-0.7)^2+(-1.18)^2$$

$$A=\sqrt{0.7^2+1.18^2} = 1.37\,\mathrm{rad}$$

$$\frac{(2)}{(1)}$$

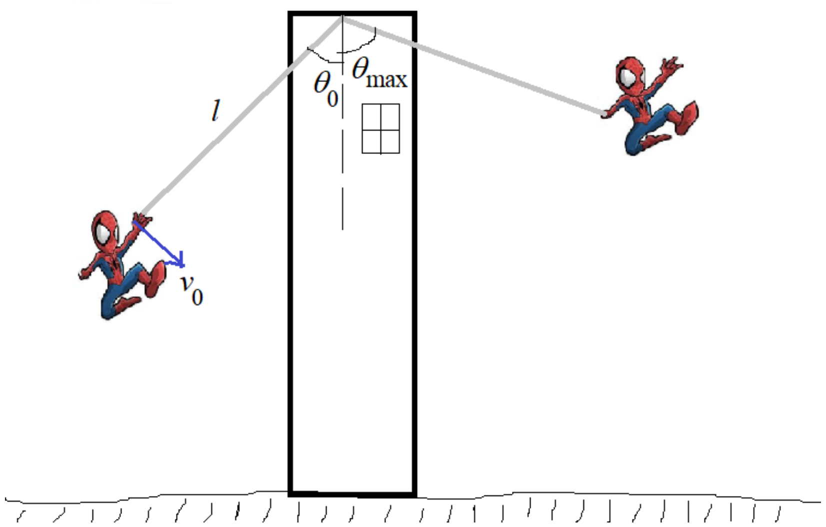
$$\frac{A\sin\varphi_0}{A\cos\varphi_0} = \frac{-1.18}{-0.7}$$

$$\varphi_0 = \tan^{-1}\left(\frac{-1.18}{-0.7}\right) = \pi + \tan^{-1}\left(\frac{1.18}{0.7}\right) = 4.2\,\mathrm{rad}$$

And so, we got,

$$\theta(t) = 1.37\cos(0.57t+4.2)$$

A.2 Pendula



(b) What are his angular velocity, angular acceleration, speed, tangential acceleration, and centripetal acceleration, as a function of time?

Well,

$$\theta(t) = 1.37\cos(0.57t + 4.2)$$

$$\omega(t) = \frac{d\theta}{dt} = -0.78\sin(0.57t + 4.2)$$

$$\alpha(t) = \frac{d^2\theta}{dt^2} = -0.45\cos(0.57t + 4.2)$$

$$v(t) = \omega(t) \cdot l = -23.4\sin(0.57t + 4.2)$$

$$a_s(t) = \alpha(t) \cdot l = -13.5\cos(0.57t + 4.2)$$

$$a_c(t) = \frac{v(t)^2}{l} = 18.3\sin^2(0.57t + 4.2)$$

(c) To what maximum angle will he rise? What is his maximum speed?

$$\theta_{max} = 1.37\,\text{rad} = 78^\circ$$

$$v_{max} = 23.4\,\text{m/s}$$

(d) When will he get to the apex?

This is when

$$1.37 = 1.37\cos(0.57t + 4.2)$$

$$1 = \cos(0.57t + 4.2)$$

$$0.57t + 4.2 = 0, 2\pi, 4\pi, etc.$$

$$t = \frac{2\pi - 4.2}{0.57} = 3.7\,\text{s}$$

(e) What would his period be if he just keeps swinging back and forth?

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{0.57} = 11\text{s}$$